Topic: Half-angle identities

Question: Given the inequality, which of the following is true?

$$-\frac{27\pi}{11} < \theta < -\frac{23\pi}{11}$$

Answer choices:

A $\cos\frac{\theta}{2} > 0$ and $\sin\frac{\theta}{2} > 0$

B $\cos\frac{\theta}{2} > 0$ and $\sin\frac{\theta}{2} < 0$

C $\cos\frac{\theta}{2} < 0$ and $\sin\frac{\theta}{2} > 0$

D $\cos\frac{\theta}{2} < 0$ and $\sin\frac{\theta}{2} < 0$

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Solution: C

We're given that

$$-\frac{27\pi}{11} < \theta < -\frac{23\pi}{11}$$

Dividing through by 2:

$$-\frac{27\pi}{22} < \frac{\theta}{2} < -\frac{23\pi}{22}$$

Now $27 = 44 - 17$, so

$$-\frac{27\pi}{22} = -\frac{(44-17)\pi}{22} = -\left(\frac{44\pi}{22} - \frac{17\pi}{22}\right) = -2\pi + \frac{17\pi}{22}$$

Also, $23 = 44 - 21$, so

$$-\frac{23\pi}{22} = -\frac{(44-21)\pi}{22} = -\left(\frac{44\pi}{22} - \frac{21\pi}{22}\right) = -2\pi + \frac{21\pi}{22}$$

Substituting these results:

$$-2\pi + \frac{17\pi}{22} < \frac{\theta}{2} < -2\pi + \frac{21\pi}{22}$$

Now

$$\frac{17\pi}{22} = \frac{34\pi}{44} > \frac{33\pi}{44} = \frac{3\pi}{4}$$

Turning this around, we get

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$$\frac{3\pi}{4} < \frac{17\pi}{22}$$

Subtracting 2π from both sides of this inequality, we have

$$-2\pi + \frac{3\pi}{4} < -2\pi + \frac{17\pi}{22}$$

Also,

$$\frac{21\pi}{22} < \frac{22\pi}{22} = \pi$$

Subtracting 2π from both sides of this inequality yields

$$-2\pi + \frac{21\pi}{22} < -2\pi + \pi$$

Substituting these results, we obtain

$$-2\pi + \frac{3\pi}{4} < -2\pi + \frac{17\pi}{22} < \frac{\theta}{2} < -2\pi + \frac{21\pi}{22} < -2\pi + \pi$$

Thus $\theta/2$ is in the interval

$$\left(-2\pi + \frac{3\pi}{4}, -2\pi + \pi\right)$$

This tells us that $\theta/2$ is coterminal with an angle α in the interval

$$\left(\frac{3\pi}{4}, \pi\right)$$

so $\cos(\theta/2)$ and $\sin(\theta/2)$ are equal to $\cos\alpha$ and $\sin\alpha$, respectively.

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Note that α is in the second quadrant, and recall that the cosine of every angle in the second quadrant is negative and the sine of every angle in that quadrant is positive. Therefore,

$$\cos\frac{\theta}{2} < 0 \text{ and } \sin\frac{\theta}{2} > 0$$

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Topic: Half-angle identities

Question: If θ is the angle in the interval $(3\pi/2, 2\pi)$ with $\sin\theta = -2/3$, what are the values of $\cos(\theta/2)$ and $\sin(\theta/2)$?

Answer choices:

A $\cos\frac{\theta}{2} = -\sqrt{\frac{3-\sqrt{7}}{6}}$ and $\sin\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{7}}{6}}$

B $\cos\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{7}}{6}}$ and $\sin\frac{\theta}{2} = -\sqrt{\frac{3-\sqrt{7}}{6}}$

C $\cos\frac{\theta}{2} = -\sqrt{\frac{3+\sqrt{5}}{6}}$ and $\sin\frac{\theta}{2} = \sqrt{\frac{3-\sqrt{5}}{6}}$

D $\cos\frac{\theta}{2} = \sqrt{\frac{3-\sqrt{5}}{6}}$ and $\sin\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{5}}{6}}$

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Solution: C

We first have to determine the value of $\cos\theta$.

By the basic Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$

Therefore,

$$\cos^2\theta = 1 - \sin^2\theta$$

Substituting $-2/3$ for $\sin\theta$, we get

$$\cos^2\theta = 1 - \left(-\frac{2}{3}\right)^2 = 1 - \left(\frac{4}{9}\right) = \frac{5}{9}$$

Now

$$\frac{3\pi}{2} < \theta < 2\pi$$

so θ is in the fourth quadrant, and $\cos\theta$ is positive:

$$\cos\theta = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

By the half-angle identity for cosine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{5}}{3}}{2}} = \pm\sqrt{\frac{\frac{3}{3}+\frac{\sqrt{5}}{3}}{2}} = \pm\sqrt{\frac{3+\sqrt{5}}{6}}$$

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By the half-angle identity for sine,

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}=\pm\sqrt{\frac{1-\frac{\sqrt{5}}{3}}{2}}=\pm\sqrt{\frac{\frac{3}{3}-\frac{\sqrt{5}}{3}}{2}}=\pm\sqrt{\frac{3-\sqrt{5}}{6}}$$

Recall that θ is in the interval $(3\pi/2,2\pi)$, which is a subinterval of $[0,2\pi)$, so $\theta/2$ is in the second quadrant. Therefore, $\cos(\theta/2)$ is negative and $\sin(\theta/2)$ is positive:

$$\cos\frac{\theta}{2}=-\sqrt{\frac{3+\sqrt{5}}{6}}$$

$$\sin\frac{\theta}{2}=\sqrt{\frac{3-\sqrt{5}}{6}}$$

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Topic: Half-angle identities

Question: If θ is the angle in the interval $(17\pi/2, 9\pi)$ with $\cos\theta = -3/7$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Answer choices:

A $\sin\frac{\theta}{2} = -\sqrt{\frac{1}{7}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{6}{7}}$

B $\sin\frac{\theta}{2} = \sqrt{\frac{5}{7}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{2}{7}}$

C $\sin\frac{\theta}{2} = \sqrt{\frac{3}{7}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{4}{7}}$

D $\sin\frac{\theta}{2} = \sqrt{\frac{2}{7}}$ and $\cos\frac{\theta}{2} = -\sqrt{\frac{5}{7}}$

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Solution: B

By the half-angle identity for cosine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Substituting $-3/7$ for $\cos\theta$ gives

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\left(-\frac{3}{7}\right)}{2}} = \pm\sqrt{\frac{\frac{7(1)+1(-3)}{7}}{2}} = \pm\sqrt{\frac{7-3}{14}} = \pm\sqrt{\frac{4}{14}} = \pm\sqrt{\frac{2}{7}}$$

By the half-angle identity for sine,

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Substituting $-3/7$ for $\cos\theta$ gives

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\left(-\frac{3}{7}\right)}{2}} = \pm\sqrt{\frac{\frac{7(1)+1(3)}{7}}{2}} = \pm\sqrt{\frac{7+3}{14}} = \pm\sqrt{\frac{10}{14}} = \pm\sqrt{\frac{5}{7}}$$

We were given that

$$\frac{17\pi}{2} < \theta < 9\pi$$

Dividing through by 2:

$$\frac{17\pi}{4} < \frac{\theta}{2} < \frac{9\pi}{2}$$

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Note that

$$\frac{17\pi}{4} = \frac{16\pi}{4} + \frac{\pi}{4} = 4\pi + \frac{\pi}{4}$$

and

$$\frac{9\pi}{2} = \frac{8\pi}{2} + \frac{\pi}{2} = 4\pi + \frac{\pi}{2}$$

Substituting these results, we have

$$4\pi + \frac{\pi}{4} < \frac{\theta}{2} < 4\pi + \frac{\pi}{2}$$

Since 4π is an integer multiple of 2π, an angle of 4π radians lies on the positive x-axis, so any angle between $4\pi + (\pi/4)$ and $4\pi + (\pi/2)$ lies strictly "between" the positive x-axis and the positive y-axis. Therefore, $\theta/2$ is in the first quadrant, hence both $\sin(\theta/2)$ and $\cos(\theta/2)$ are positive:

$$\sin\frac{\theta}{2} = \sqrt{\frac{5}{7}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{2}{7}}$$

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